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ITERATIVE METHODS FOR THE NUMERICAL SOLUTION OF
SECOND ORDER ELLIPTIC EQUATIONS WITH LARGE FIRST ORDER TERMS

(NASA-CR-185757) ITERATIVE METHODS FOR THE NUMERICAL SOLUTION OF SECOND ORDER ELLIPTIC EQUATIONS WITH LARGE FIRST ORDER TERMS (ICASE) 27 p

N89-71429

Unclas
00/64 0224318

John C. Strikwerda

Report Number 79-5
February 8, 1979

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia

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UNIVERSITIES SPACE USRA RESEARCH ASSOCIATION

Errata for Report 79-5

p. 3, middle, should be:

$$q'(0) \gg 1$$

p. 4, line 7:

$$u_{i-1} + u_i - \frac{2q'_{i-\frac{1}{2}}}{Rh}(u_{i-1} - u_i) > 2u_i$$

p. 12, line 3 from bottom:

$$(\omega^*)^{-1} \leq \;\; \ldots\ldots.$$

p. 14, line 4 from bottom:

$$(\omega^*)^{-1} \leq \;\; \ldots\ldots.$$

Reference [2] should be:

Concus, P. and Golub, G. H., *A generalized conjugate gradient method for the numerical solution of elliptic partial differential equations*, Proceedings of the 2nd International Symposium on Computing Methods in Applied Sciences and Engineering, (IRIA, Paris, December 1975); Lecture Notes in Economics and Mathematical Systems, Vol. 134, eds. R. Glowinski and J. L. Lions, Springer-Verlag, 1976.

ITERATIVE METHODS FOR THE NUMERICAL SOLUTION OF SECOND ORDER ELLIPTIC EQUATIONS WITH LARGE FIRST ORDER TERMS

John Strikwerda

Institute for Computer Applications in Science and Engineering

ABSTRACT

This paper presents iterative methods for the numerical solution of second order elliptic equations whose first order terms have coefficients that are orders of magnitude larger than those of the second order terms. Such equations arise in singular perturbation problems and also in the numerical grid generation technique of Mastin and Thompson. These equations exhibit boundary layer phenomena which usually require an unevenly spaced grid for their numerical solution. The methods are similar to successive overrelaxation, but have the advantage of not requiring the user to supply a parameter. The methods are shown to be stable even for variable coefficients by using the theory of pseudo-translation operators developed by Vaillancourt. Numerical results are presented and discussed.

This report was prepared as a result of work performed under NASA Contract No. NAS1-14101 at ICASE, NASA Langley Research Center, Hampton, VA 23665.

I. Introduction

Consider the elliptic equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = G(x,y) \tag{1.1}$$

defined in a domain Ω in $\mathbb{R}^2$. The coefficients are assumed to be smooth, slowly varying functions of the independent variables (x,y), and also

$$ac - b^2 \geq \delta > 0$$

on Ω. If L is a reference length for Ω, such as the diameter, then we define the Reynolds number for equation (1.1) as

$$R(x,y) = \frac{L\sqrt{d^2 + e^2}}{\frac{1}{2}(a + c)} \; . \tag{1.2}$$

In this paper we will consider elliptic equations for which the Reynolds number is large, on the order of a thousand at least. We will also assume that the coefficient $f(x,y)$ is of the same, or less, order of magnitude as the coefficients $d(x,y)$ and $e(x,y)$, and that the coefficients $a(x,y)$ and $c(x,y)$ are of the same order of magnitude with respect to the Reynolds number. Such equations as these arise frequently in applications, usually as singular perturbation problems.

The methods presented in this paper are designed for the numerical solution of elliptic equations with large Reynolds number. They formally resemble successive-over-relaxation (SOR) and they will be referred to as SRR - successive relaxation for large Reynolds number.

Unlike SOR which requires apriori knowledge of the iteration parameter, in SRR the iteration parameter is chosen at each grid point according to a formula derived from a stability criterion. This formula for the case of the frequently encountered five-point difference operator is derived in Section V. Computational results are described in the last section and the notable features of SRR are presented. The author is currently working on a proof for the convergence of the SRR method.

Iterative numerical methods for elliptic equations with sizeable lower order terms have been studied by several other authors; we mention only Concus, Golub, and O'Leary [2] and Widlund [10]. They, however, do not consider problems with boundary layers and nonuniform grids which are essential features of SRR.

II. Boundary layers

The solutions of Dirichlet problems for elliptic equations with a large Reynolds number frequently have boundary layers. Boundary layers are regions near the boundary where the solution has very large gradients and they are located at those boundary points where

$$a(x,y)(d(x,y)n_x + e(x,y)n_y) > 0 , \tag{2.1}$$

and (n_x, n_y) are the direction cosines for the interior normal at (x,y) . The condition (2.1) can be established by the method of perturbation expansions (see Nayfeh [4], Van Dyke [9]).

In the numerical solution of such problems a nonuniform grid is frequently employed to place more grid points in the boundary layer region. To study some effects of the grid on the solution we consider the one-dimensional equation

$$u_{xx} + Ru_x = 0 \quad \text{on } 0 \leq x \leq 1 \tag{2.2}$$

with the boundary data

$$u(0) = 0 \ , \ u(1) = 1 \ .$$

There is a boundary layer at $x = 0$, and to resolve it we introduce the coordinate transformation

$$q = q(x) \ , \ q(0) = 0 \ , \ q(1) = 1$$

where

$$q'(x) > 0 \quad \text{and} \quad q'(0) << 1 \ .$$

Equation (1.4) then becomes

$$(q'u_q)_q + Ru_q = 0 \ ,$$

and this equation is approximated by the difference equations

$$\begin{aligned} &h^{-2}(q'_{i+\frac{1}{2}}(u_{i+1}-u_i) - q'_{i-\frac{1}{2}}(u_i - u_{i-1})) \\ &+ \tfrac{1}{2}Rh^{-1}(u_{i+1}-u_{i-1}) = 0, \qquad i = 1, 2, \cdots, N-1 \ , \end{aligned} \tag{2.3}$$

with

$$u_0 = 0, \quad u_N = 1 \ .$$

The solution to equation (2.2) is a monotone increasing function and we will now examine equation (2.3) to see when its solution is also monotone. We rewrite equation (2.3) as

$$u_{i+1} + u_i + \left(\frac{2q'_{i+\frac{1}{2}}}{Rh} \right) (u_{i+1} - u_i)$$

$$= u_i + u_{i-1} + \left(\frac{2q'_{i-\frac{1}{2}}}{Rh} \right) (u_i - u_{i-1}) \ .$$

Suppose u_{i+1} is greater than u_i, then we have

$$u_{i-1} + u_i + \frac{2q'_{i-\frac{1}{2}}}{Rh} (u_{i-1} - u_i) > 2u_i \ ,$$

or equivalently

$$(u_{i-1} - u_i) \left(1 - \frac{2q'_{i-\frac{1}{2}}}{Rh} \right) > 0 \ .$$

If the solution to equation (2.3) is monotone then u_i will be greater than u_{i-1}, and this requires that the second term in the above inequality be negative. This shows that a necessary condition for the solution to equation (2.3) to be monotone is that

$$\frac{Rh}{q'_{i+\frac{1}{2}}} \leq 2 \quad \text{for } i = 0, 1, \cdots, N-1 \ . \tag{2.4}$$

The quantity on the left side of inequality (2.4) is called the cell Reynolds number. Note that the condition is essentially

$$R(x_{i+1} - x_i) \leq 2 \ , \tag{2.4'}$$

and it imposes a restriction on the grid spacing.

In computation it is seen that if inequality (2.4) is violated at grid points away from the boundary layer then the resulting oscillations are not large and do not severely affect the accuracy of the solution. However, if the inequality is violated in the boundary layer region then the oscillations can be very large and will severely affect the accuracy of the solution.

It should be mentioned that another difference scheme might lead to a slightly different condition than condition (2.4) but it would still be a restriction on the grid spacing similar to the restriction (2.4'). It is also important to note that the cell Reynolds number condition is a statement about the solution of the difference equations and is independent of the solution procedure. For two-dimensional problems and problems with variable coefficients the cell Reynolds number condition remains approximately valid in the neighborhood of the boundary layer. For another discussion of the cell Reynolds number condition see Roache [5], and for a finite element approach to this subject see Christie et.al. [1].

III. The SRR method, an example

Before introducing the method in general, we will consider as an illustration the equation

$$u_{xx} + u_{yy} + R_1 u_x + R_2 u_y = 0 \tag{3.1}$$

on the square

$$0 \leq x \leq 1 \;,\; 0 \leq y \leq 1 \;,$$

with u specified on the boundary. To resolve the boundary layers we introduce a change of coordinates given by

$$q = q(x) \quad , \quad p = p(y)$$

where $q(x)$ and $p(y)$ are smooth, strictly increasing functions and

$$q(0) = p(0) = 0, \quad q(1) = p(1) = 1 \ .$$

Equation (3.1) then becomes, in the new coordinates,

$$q'(q'u_q)_q + p'(p'u_p)_p + R_1 q' u_q + R_2 p' u_p = 0 \ ,$$

where

$$q' = \frac{dq}{dx} \quad \text{and} \quad p' = \frac{dp}{dy} \quad .$$

This equation can then be replaced by a difference approximation using a uniform grid in the (q, p) unit square. We write this difference approximation only as

$$m_{ij}^{1,0} u_{i+1j} + m_{ij}^{-1,0} u_{i-1j} + m_{ij}^{0,1} u_{ij+1} + m_{ij}^{0,-1} u_{ij-1}$$
$$- (m_{ij}^{1,0} + m_{ij}^{-1,0} + m_{ij}^{0,1} + m_{ij}^{0,-1}) u_{ij} = 0 \quad ,$$

where the coefficients $m_{i,j}^{a,b}$ depend on the precise form of differencing that is used.

Using the natural ordering of points and immediate replacement this system of difference equations can be solved by the following algorithm

$$u_{ij}^{n+1} = u_{ij}^{n} + \omega_{ij} \{ m_{ij}^{1,0} u_{i+1j}^{n} + m_{ij}^{-1,0} u_{i-1j}^{n+1}$$
$$+ m_{ij}^{0,1} u_{ij+1}^{n} + m_{ij}^{0,-1} u_{ij-1}^{n+1} \tag{3.3}$$
$$- (m_{ij}^{1,0} + m_{ij}^{-1,0} + m_{ij}^{0,1} + m_{ij}^{0,-1}) u_{ij}^{n} \} \ .$$

The iteration parameter ω_{ij} is given by

$$\omega_{ij} = \frac{2}{\tilde{m} + \sqrt{\tilde{m}} \sqrt{\frac{(m^{1,0} - m^{-1,0})^2}{(m^{1,0} + m^{-1,0})} + \frac{(m^{0,1} - m^{0,-1})^2}{(m^{0,1} + m^{0,-1})}}} \tag{3.4}$$

where $\tilde{m} = m^{1,0} + m^{-1,0} + m^{0,1} + m^{0,-1}$ and the subscripts (i,j) have been left off the m's for convenience.

This example will be discussed at more length in Section VI and the derivation of the expression for ω_{ij} will be given in Section V. For now we only point out that for the particular case whose results are given in the first part of Table I the number of interations required for convergence is nearly constant, independent of R for values of R between 6,000 and 80,000.

IV. The SRR method

We now present the SRR method in detail. We begin with equation (1.1) and transform coordinates to the independent variables $(\bar{x},\bar{y})$ in order to improve the resolution of the boundary layers. This transformed equation again has the form of equation (1.1). We will assume for simplicity of exposition that $\bar{\Omega}$, the image of Ω in the $(\bar{x},\bar{y})$ coordinates, is the unit square. On $\bar{\Omega}$ we take a uniform grid with points

$$X_\alpha = (\alpha_1 h_1, \alpha_2 h_2)$$

indexed by the multi-index $\alpha = (\alpha_1, \alpha_2)$. The α_i are integers with

$$0 \le \alpha_i \le h_i^{-1}$$

where the quantities h_i^{-1} are also integers.

The difference approximation can then be written as

$$\sum_{|\beta| \le b} M_{\alpha\beta} u_{\alpha+\beta} = G_\alpha \simeq G(\bar{x}_\alpha, \bar{y}_\alpha) \tag{4.1}$$

for all multi-indices α with $X_\alpha \in \bar{\Omega}$. The norm of a multi-index is given by

$$|\beta| = |(\beta_1, \beta_2)| = |\beta_1| + |\beta_2| .$$

The class of iterative methods we discuss here are given in general by

$$u_\alpha^{n+1} = u_\alpha^n + \omega_\alpha \left(\sum_{|\beta| \le b} M_{\alpha\beta}^0 u_{\alpha+\beta}^n + \sum_{|\beta| \le b} M_{\alpha\beta}^1 u_{\alpha+\beta}^{n+1} \right) \tag{4.2}$$

where

$$M_{\alpha\beta}^0 + M_{\alpha\beta}^1 = M_{\alpha\beta} .$$

To determine the iteration parameter ω_α for this scheme, we will study the symbol of the scheme. The symbol for the scheme (4.2) is defined as

$$p(X_\alpha, \xi) = \frac{\omega_\alpha^{-1} + \sum_{|\beta| \le b} M_{\alpha\beta}^0 e^{i\xi\cdot\beta}}{\omega_\alpha^{-1} - \sum_{|\beta| \le b} M_{\alpha\beta}^1 e^{i\xi\cdot\beta}}$$

where $\xi = (\xi_1, \xi_2)$ with $|\xi_i| \le \pi$. The iteration parameter ω_α is chosen so that

$$|p(X_\alpha, \xi)| \le 1, \quad \text{for} \quad |\xi_i| \le \pi . \tag{4.3}$$

The definition of the symbol of the iteration scheme is in accordance with the theory of pseudo-translation operators developed by Vaillancourt [7]. The condition (4.3) is motivated by the Lax-Nirenberg theorem (Lax-Nirenberg [3], Vaillancourt [8]), which guarantees stability for the iteration scheme when applied for $\bar{\Omega} = \mathbb{R}^2$.

For the remainder of the paper we make the following consistency assumption.

<u>Assumption 4.1.</u> The difference approximation (4.1) satisfies

$$- \mathcal{Re} \sum_{|\beta| \le b} M_{\alpha\beta} e^{i\xi \cdot \beta} \ge c(|\xi_1|^2 + |\xi_2|^2)$$

for $|\xi_i| \le \pi$ and some positive constant c.

We now obtain an expression for ω_α. Define the symbols

$$m^1 = m^1(x,\xi) = \sum_{|\beta| \le b} M^1_{\alpha\beta} e^{i\xi\cdot\beta} ,$$

$$m^0 = m^0(x,\xi) = \sum_{|\beta| \le b} M^0_{\alpha\beta} e^{i\xi\cdot\beta} ,$$

and

$$m = m(x,\xi) = m^0(x,\xi) + m^1(x,\xi) .$$

From $|p| \le 1$ we have

$$|\omega^{-1} + m^0|^2 \le |\omega^{-1} - m^1|^2 ,$$

or equivalently

$$2\mathcal{R}e\, m\, \omega^{-1} \le |m^1|^2 - |m^0|^2 .$$

From the consistency assumption we have

$$\omega^{-1} \ge \frac{|m^0|^2 - |m^1|^2}{-2\mathcal{R}e\, m} .$$

Define ω_α^* by

$$(\omega_\alpha^*)^{-1} = \sup_{|\xi_i| \le \pi} \frac{|m^0|^2 - |m^1|^2}{-2\mathcal{R}e\, m} . \tag{4.4}$$

For ω_α in the interval $[0, \omega_\alpha^*]$ the iterative method (4.2) will be stable. Moreover, as shown in the last section of this paper, the convergence rate of this scheme is optimal for ω_α equal to ω_α^*, at least for the examples considered there.

Note that if $|m^1|$ is larger than $|m^0|$ for all values of ξ, then the scheme is unconditionally stable in the sense that any positive value of ω_α will satisfy inequality (4.3). We will not consider such schemes here.

V. Computation of ω^* for special cases

We first compute ω^* for the case in which the iteration operator has a five-point stencil given by

$$u_{ij}^{n+1} = u_{ij}^{n} + \omega_{ij} \{a_{ij} u_{i+1j}^{n} + b_{ij} u_{i-1j}^{n+1} + c_{ij} u_{ij+1}^{n} + d_{ij} u_{ij-1}^{n+1} - (a_{ij} + b_{ij} + c_{ij} + d_{ij}) u_{ij}^{n} \} . \tag{5.1}$$

Equation (5.1) is of the same form as equation (3.3). As in the previous section we have

$$m^0 = ae^{i\theta} + ce^{i\phi} - (a+b+c+d)$$

$$m^1 = be^{-i\theta} + de^{-i\phi}$$

where we have dropped the subscripts (i,j) and $\xi = (\theta,\phi)$. Note that Assumption 4.1 is satisfied when $a+b$ and $c+d$ are positive,

$$\begin{aligned} -\mathcal{Re}\, m = -\mathcal{Re}\, (m^0 + m^1) &= (a+b)(1-\cos\theta) + (c+d)(1-\cos\phi) \\ &= 2(a+b)\sin^2 \tfrac{1}{2}\theta + 2(c+d)\sin^2 \tfrac{1}{2}\phi \quad . \end{aligned}$$

We will use formula (4.4) to compute ω^*.

$$\begin{aligned} |m^0|^2 - |m^1|^2 = {} & (a+b+c+d)(-\mathcal{Re}\, m) \\ & + 2(a+b)(a-b-c+d)\sin^2 \tfrac{1}{2}\theta \\ & + 2(c+d)(c-d-a+b)\sin^2 \tfrac{1}{2}\phi \\ & + 4((a+b)(c-d) + (c+d)(a-b))\sin \tfrac{1}{2}\theta \, \sin \tfrac{1}{2}\phi \, \cos \tfrac{1}{2}(\theta - \phi) \quad . \end{aligned}$$

Let

$$A = 2(a+b)(a-b-c+d)\ ,$$

$$B = 2((a+b)(c-d)+(c+d)(a-b))\ ,$$

and

$$C = 2(c+d)(c-d-a+b)\ .$$

Then

$$|m^0|^2 - |m^1|^2 \le (a+b+c+d)(-\mathcal{R}e\, m)$$

$$+ (A+|B|r)\sin^2 \tfrac{1}{2}\theta + (C+|B|r^{-1})\sin^2 \tfrac{1}{2}\phi\ ,$$

where r is an arbitrary positive number. If r is chosen so that

$$\frac{A+|B|r}{a+b} = \frac{C+|B|r^{-1}}{c+d}\ ,$$

then

$$|m^0|^2 - |m^1|^2 \le (a+b+c+d)(-\mathcal{R}e\, m)$$

$$+ \sqrt{(a+b+c+d)\left(\frac{(a-b)^2}{a+b} + \frac{(c-d)^2}{c+d}\right)}\,(-\mathcal{R}e\, m)\ .$$

This implies that

$$(\omega^*)^{-1} \ge \frac{a+b+c+d}{2} + \frac{1}{2}\sqrt{(a+b+c+d)\left(\frac{(a-b)^2}{a+b} + \frac{(c-d)^2}{c+d}\right)}\ .$$

Moreover the above inequalities are sharp as can be seen by taking

$$\theta = \phi/r \longrightarrow 0\ .$$

Therefore

$$\omega^* = \frac{2}{a+b+c+d + \sqrt{(a+b+c+d)\left(\frac{(a-b)^2}{a+b} + \frac{(c-d)^2}{c+d}\right)}} \quad (5.2)$$

The reader is reminded that the coefficients $a, b, c,$ and d are all variable functions of the grid points and therefore ω^* is also a function of the grid point.

We now consider the iterative method obtained from the checkerboard ordering. We will show that in this case ω^* is also given by formula (5.2). Analogous to equation (5.1) we have

$$\begin{aligned} u_{ij}^{n+1} = u_{ij}^{n} + \omega_{ij}\{ & a_{ij} u_{i+1j}^{n+\varepsilon} + b_{ij} u_{i-1j}^{n+\varepsilon} \\ & c_{ij} u_{ij+1}^{n+\varepsilon} + d_{ij} u_{ij-1}^{n+\varepsilon} \\ & - (a_{ij} + b_{ij} + c_{ij} + d_{ij}) u_{ij}^{n}\} \end{aligned} \quad (5.3)$$

where $\varepsilon = 0$ for $i+j$ even and $\varepsilon = 1$ for $i+j$ odd. To obtain the expression for ω_{ij}^* one sets

$$u_{k\ell}^{n} = z^{2n+\varepsilon} e^{ik\theta} e^{i\ell\phi} ,$$

obtaining the equation

$$\begin{aligned} & \frac{z}{\omega} - \frac{1}{z}\left(\frac{1}{\omega} - (a+b+c+d)\right) \\ & = (a+b)\cos\theta + (c+d)\cos\phi \\ & + i(a-b)\sin\theta + i(c-d)\sin\phi . \end{aligned} \quad (5.4)$$

The value of ω^* is determined by the requirement that for ω less than or equal to ω^* the modulus of z is less than unity. Rather than to proceed with this calculation an alternative approach is to notice that if for the natural ordering one sets

$$u^n_{k\ell} = z^{2n+k+\ell} e^{ik\theta} e^{i\ell\phi}$$

then the resulting formula for z is the same as equation (5.4). This shows that the expression for ω^* is the same for both orderings.

For the iterative method whose formula is

$$\begin{aligned} u^{n+1}_{ij} = u^n_{ij} + \omega_{ij}\{ & A_{ij}(u^n_{i+1j} - 2u^n_{ij} + u^{n+1}_{i-1j}) \\ & + B_{ij}(u^n_{i+1\,j+1} - u^{n+1}_{i+1\,j-1} - u^n_{i-1\,j+1} + u^{n+1}_{i-1\,j-1}) \\ & + C_{ij}(u^n_{ij+1} - 2u^n_{ij} + u^{n+1}_{ij-1}) \\ & + D_{ij}(u^n_{i+1j} - u^{n+1}_{i-1j}) \\ & + E_{ij}(u^n_{ij+1} - u^{n+1}_{ij-1})\} , \end{aligned}$$

an estimate for ω^* is given by

$$\begin{aligned} (\omega^*)^{-1} \geq \tfrac{1}{2}\{ & A + C + D + E \\ & + \frac{2(1+\sqrt{2})(A|E| + C|D|) + 4|B|(2\sqrt{2}\,|D| + |D+E|)}{A + C - \sqrt{(A-C)^2 + 16B^2}} \end{aligned}$$

This estimate of ω^* has been used to numerically solve the elliptic equations that result from the grid generation technique of Thompson et. al. [6]. In the case that the grid has a high degree of stretching the equations have a large Reynolds number. Although no results on this are presented in this paper, it has performed very satisfactorily in many computations.

VI. Computational Results

The SRR method presented in this paper has been tested on the differential equation

$$u_{xx} + u_{yy} + Ru_x = 0 \qquad 0 \le x,y \le 1 ,$$

with several boundary conditions. The coordinates were transformed by the mappings

$$\begin{aligned} x &= x_0' q + (1 - x_0')q^4 \\ y &= \tfrac{1}{2} + (1 - 7c)(p - \tfrac{1}{2}) + 40\,c\,(p - \tfrac{1}{2})^3 - 48\,c\,(p - \tfrac{1}{2})^5 \end{aligned} \tag{6.1}$$

where x_0' and c are parameters. Note that

$$x_0' = \frac{dx}{dq}(0)$$

and

$$y_0' = \frac{dy}{dp}(0) = \frac{dy}{dp}(1) = 1 + 8c ,$$

and also

$$\frac{d^2x}{dq^2}(0) = \frac{d^3x}{dq^3}(0) = \frac{d^2y}{dp^2}(0) = \frac{d^2y}{dp^2}(1) = 0 .$$

Employing a uniform $(N+1)\times(N+1)$ grid in the (q,p) plane the difference approximation is given by

$$q_i'(q_{i+\frac{1}{2}}'(u_{i+1j}-u_{ij})-q_{i-\frac{1}{2}}'(u_{ij}-u_{i-1j}))h^{-2}$$

$$p_j'(p_{j+\frac{1}{2}}'(u_{ij+1}-u_{ij})-p_{j-\frac{1}{2}}'(u_{ij}-u_{ij-1}))h^{-2}$$

$$+\tfrac{1}{2}R\,q_i'(u_{i+1j}-u_{i-1j})h^{-1}=0$$

for $1<i,\ j<N+1$, where $h=N^{-1}$.

The first problem has the boundary condition

$$u(x,y)=|x-y|$$

for (x,y) on the boundary. An exact solution is not known for this problem; however, the solution does satisfy

$$u(x,1-y)=1-u(x,y)$$

and for R large and $\delta<y<1-\delta$

$$u(x,y)\simeq y\,e^{-xR}+(1-e^{-xR})(1-y)\ .$$

In addition to the boundary layer at $x=0$, the solution to this problem has appreciable gradients along the boundaries $y=0$ and $y=1$.

The second problem has as its solution

$$u(x,y)=\left(y(1-y)-\frac{2x}{R}\right)e^{-Rx}$$

and the boundary conditions specify that u agrees with this solution on the boundary.

The concern in this paper is with the convergence properties of the method, and not so much with the accuracy of the results. For any particular problem the accuracy of the results depend primarily on having enough grid points to resolve the boundary layer and having a coordinate transformation which places the grid points properly. The cell Reynolds numbers are a measure of the suitability of the placement of the grid points. A second consideration is the criteria to terminate the iterative method. In the following examples the iterative procedure was stopped when

$$\frac{\|u^n - u^{n-1}\|_2}{\|u^n\|_2} \leq 10^{-4} . \tag{6.2}$$

The ℓ^2 norm in the above is given by

$$\|f\|_2 = \left(h^{-2} \sum_{X_\alpha \varepsilon \bar{\Omega}} |f_\alpha|^2\right)^{\frac{1}{2}} .$$

The convergence criterion of 10^{-4} in inequality (6.2) was chosen because it was small enough to achieve satisfactory answers and yet large enough so that it was economically feasible to make the large number of runs required for testing the algorithm.

In Tables I and II are shown the results of solving problems 1 and 2, respectively, by both the checkerboard and natural orderings for different values of the grid size N, the Reynolds number R, and the coordinate transformations. In these tables the value of ω_α was always taken to be ω_α^* as given by equation (5.2).

Notice from Tables I and II that for a given grid size, coordinate transformation, and ordering the number of iterations required for convergence is essentially independent of R. However, when the cell Reynolds number at $x=0$ is near or above 2 the convergence rate becomes poorer and the solution itself becomes highly oscillatory.

Included at the end of Table I are calculations for the equation

$$u_{xx} + u_{yy} + Ru_x + \delta Ru = 0$$

for $\delta = 0.5$ and $\delta = 1.0$. The value of ω_α was the same as derived in Section V for the case $\delta = 0$. The computations indicate that stability is maintained for lower order terms of the same order as the first order terms.

Table III shows the effect of taking ω_α as a multiple of ω_α^* for the case when $N = 40$ and $R = 40,000$. The number of iterations required for convergence was least for $\omega_\alpha = \omega_\alpha^*$ and for ω_α larger than ω_α^* the method does not converge at all. Similar results were observed for other values of N and R but are not displayed.

In Table IV is shown the relationship between the number of grid points along one side of the grid, N, and the number of iterations required for convergence. For both the checkerboard ordering and natural ordering the number of iterations is proportional to N for larger values of N. This shows that ρ, the radius of convergence of the iterative scheme, satisfies

$$\rho = 1 - C/N + o(N^{-1})$$

and, from the earlier comments, C is independent of R. A similar formula holds for SOR when the iteration parameter is chosen properly (Young [11]), and this is an indication of the efficiency of the method.

More specifically, we have from the results of Table IV that for the checkerboard ordering the radius of convergence satisfies

$$\rho \cong 1 - \frac{5.8}{N}$$

and for natural ordering

$$\rho \cong 1 - \frac{4.6}{N} \ .$$

By comparison, using SOR to solve Laplace's equation on the unit square with a uniform grid the radius of convergence satisfies

$$\rho \cong 1 - \frac{2\pi}{N} \ .$$

This shows that using SRR for elliptic equations with large Reynolds numbers is about as efficient as using SOR for elliptic equations with very low Reynolds numbers.

VII. Conclusion

The SRR method introduced in this paper is a stable, efficient algorithm for the numerical solution of elliptic equations with large Reynolds number. The formula for the iteration parameter given in Section V for the five-point schemes gives convergence rates that are essentially independent of the Reynolds number over a wide range of values.

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TABLE 1. Results for Problem 1.

Grid Parameters		Reynolds No. (thousands)	Iterations Checkerboard Ordering	Iterations Natural Ordering	Cell Reynolds No. at $x = 0$
$N = 40$		6	70	86	.15
$x'_0 = 10^{-3}$		10	70	87	.25
$y'_0 = 0$		30	71	89	.75
		60	72	90	1.5
		80	72	90	2.0
		90	74	122	2.25
		100	108	130	2.5
$N = 40$		80	107	120	.20
$x'_0 = 10^{-4}$		100	109	125	.25
$y'_0 = 0$		300	102	119	.75
		600	115	130	1.5
$N = 40$		10	94	109	.25
$x'_0 = 10^{-3}$		30	101	118	.75
$y'_0 = .5$		60	103	120	1.5
		80	104	121	2.0
	Reversed	10		81	.25
	Natural	30		85	.75
	Ordering	60		87	1.5
		80		92	2.0
$N = 80$		100	130	165	1.3
$x'_0 = 10^{-3}$		130	130	165	1.6
$y'_0 = 0$		160	130	165	2.0
		180	131	165	2.3
$N = 40$	$\delta = .5$	10	101		.25
$x'_0 = 10^{-3}$		30	112		.75
$y'_0 = .5$		60	115		1.5
		80	116		2.0
	$\delta = 1.$	10	112		.25
		30	128		.75
		60	132		1.5

TABLE 2. Results for Problem 2.

Grid Parameters	Reynolds No. (thousands)	Iterations Checkerboard Ordering	Iterations Natural Ordering	Cell Reynolds No. at $x = 0$
$N = 40$	6	47	46	.15
$x'_0 = 10^{-3}$	10	76	73	.25
$y'_0 = 0$	30	75	73	.75
	40	67	58	1.0
	60	80	76	1.5
	80	93	92	2.0
	90	98	97	2.3
	100	103	99	2.5
$N = 40$	80	89	85	.20
$x'_0 = 10^{-4}$	100	95	94	.25
$y'_0 = 0$	300	92	91	.75
	600	153	149	1.5

TABLE 3. Iterations for ω as a multiple of ω^* in Problem 1.

$R = 40{,}000$, $N = 40$, $x_0' = 10^{-3}$, $y_0' = 0$. $\omega_\alpha = \eta\omega_\alpha^*$

η	Checkerboard Ordering	Natural Ordering
.80	89	105
.90	79	97
.95	75	93
1.00	72	89
1.01	> 200	> 200
1.05	diverged	diverged

TABLE 4. Iterations as a function of N in Problem 1.

$R = 40{,}000$, $x_0' = 10^{-3}$, $y_0' = 0$, $\omega_\alpha = \omega_\alpha^*$

N	Checkerboard Ordering Iterations	Checkerboard Ordering Iterations/N	Natural Ordering Iterations	Natural Ordering Iterations/N
20	64	3.2	72	3.6
40	72	1.8	89	2.2
60	101	1.7	127	2.1
80	130	1.6	163	2.0
100	155	1.6	202	2.0